The Pricing of Embedded Options in Real Estate Lease Contracts

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Abstract. **Leases and rental agreements often have options attached or embedded in them. These options sometimes depend on a number of economic variables such as the Consumer Price Index (CPI), a real estate index and/or the value of real estate underlying the agreement. The evaluation of these options often involves the solution or approximation to a partial differential equation (PDE). This study analyzes the appropriate PDEs which model the situation where the lessee is granted an option to either purchase the property or to renew the lease at a price (rent) indexed to the CPI or some other readily measured economic variable. The PDEs that result from the usual contingent claim asset-pricing framework are derived and numerically solved using the finite difference method with absorbing boundaries. The value of an embedded option to renew a five year lease on class A office space in each of the twenty-five markets for which the National Real Estate Index reports quarterly rental data is estimated. An evaluation of the model's "Greeks" confirm that the model conforms to financial intuition which provides support for the accuracy of the estimates.**

Introduction

The explicit valuation of embedded options in financial contracts, such as the conversion option of convertible bonds or the put option held by a mortgagor, has received substantial attention in the contingent claims pricing literature. However, in real estate, little attention has been given to the pricing of explicit options, which now appear in almost all commercial lease contracts. The literature which does address lease option pricing is largely descriptive and, where pricing models are offered, tends to be simplistic to the point of inapplicability.[1]

Though lease options have not been explicitly priced, there is little doubt that these contract contingencies have value, which is necessarily reflected in the contractual rental stream. Even seemingly benign options, such as an option to renew at market rent, must have both a value to the lessee and impose a cost on the lessor. It would seem, given the extent to which option components are negotiated into leases, that both property/asset managers as well as tenants would have substantial interest in a formal and quantitative method of incorporating the option value into the income stream. The ability to explicitly identify the value of embedded options would make the negotiation of their inclusion a more straightforward and exact process.

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This study examines a number of different options that are common in commercial leases and develops pricing models that yield efficient prices for these options. First there is a brief discussion of various lease options and the stochastic properties of the underlying processes are explored. Then the Partial Differential Equations (PDEs) that model the dynamics of the options are derived and numerically approximated. A discussion of the prices and properties of the pricing model is then offered in order to demonstrate the efficacy of the models. A discussion of the applicability of the models to a broader range of lease options concludes the study.

Lease Options

Though there is a large variety of lease options, this study focuses on two types that are most common, and potentially, the most valuable. The first type is the option to renew a lease at the end of the initial lease period and the second is the option to purchase the leased space upon expiration of the lease. Though these two options are substantially different in terms of the right conveyed, their pricing is remarkably similar.

For the traditional option, an exercise, or strike price, is defined at the time the option is written and the option's value is largely a function of the dynamics of the underlying market price. In lease options, however, this is frequently not the case. Most commonly, the strike price is defined as a function of the underlying's market price, or related to the cumulative value of some index at the time of expiration. For example, lease renewal options commonly define renewal rent in one of three ways. First, the renewal rent could simply be defined as the *market* rent at the time the lease expires. Second, the renewal rent could be defined to be a fixed percentage of market rent at expiration, with 90% to 95% being the most commonly used percentages. Finally, renewal rent could be current rent grossed up by the cumulative change in some index, such as the Consumer Price Index (CPI).

Similarly, an option to purchase the leased property could define the exercise price to be market value at expiration, some defined percentage of market value or current market value grossed up by the cumulative change in some index.

The first possibility, renewal or purchase at market, is the least interesting of the three alternatives and the one to which standard option pricing methodology does not apply. The option will, by definition, be at-the-money or have an intrinsic value of zero at expiration. It, therefore, cannot take on value in the traditional sense. Yet, it does indeed have a value for which bounds can be identified but its price is a function of the negotiating skills and positions of the two parties. The value to the lessee is simply the present value of the combined cost of relocating and the locational goodwill established during the initial tenure. This amount is the most that the lessee will pay for the option and represents the upper bound on its price. The cost to the lessor is the present value of the opportunity cost imposed by the obligation to release the property to the lessee. This opportunity cost could take on various forms but would consist of at least the forgone ability to redirect the property theme or use (*i.e.*, medical building, legal building, etc.), the inability to attract a large block user because of the existing tenant and a restructuring of lease terms to shift risks from the lessor to the lessee. The present value of the perceived opportunity cost would be the lower bound of the option price. As indicated, the exact price between these two bounds cannot be mathematically defined because it is a function of the negotiating skills or positions of the two parties.

Of somewhat greater interest is an option where the exercise price is a defined percentage of market at the expiration of the lease. By definition, the option will be in-the-money at expiration by the defined fractional amount of the then current market rent, or price, depending on whether the option is an option to renew or an option to purchase. The value of this option is simply the present value of the defined fractional amount of expected market rent or price.

The typical pricing problem for a call option requires a model that will determine the probability that the market price will exceed the exercise price at expiration and also identify the expected market price, given that it is greater than the exercise price. In this case, the exercise price is defined to be less than the market price at expiration no matter what its value, so the in-the-money probability is one. Therefore, an appropriate pricing model must only identify the expected value of the market price at the expiration of the lease term.

In order to identify the expected future value of market rent or price, a stochastic process must be assumed. Since the value of income-producing real estate is a direct function of the expected rental stream, then it is easily assumed that both rent and price follow the same stochastic process. A standard assumption for investment assets is that their market prices follow Geometric Brownian Motion (GBM). However, other stochastic processes are possible and some of these possibilities are explored later. The assumption of GBM implies that real estate prices and rents have a lognormal distribution and by the properties of a lognormal distribution the expected value of real estate prices or rents (R) at expiration time T is:

$$E(R_T) = R_0 e^{\mu T + \sigma^2 T/2}, \tag{1}$$

where μ is the constant expected rate of return on real estate and σ is the standard deviation of returns. The value of the call option (O) held by the lessee is then,

$$O = (1 - p) R_0 e^{(\mu - r)T + \sigma^2 T/2}, \tag{2}$$

where p is the fractional proportion of market price or rent and r is the risk-free interest rate.

It is obvious from Equation (2) that the value of the option to renew or purchase at some fraction of market will vary between geographical markets according to the level and volatility of current prices or rents. For example, the option would be less valuable when attached to a lease on office space in the Houston or Denver markets than it would for office space in Manhattan due to the difference in rent levels. It would also be relatively less valuable when attached to a lease in a stable market, such as the northern New Jersey office market, than it would be in a volatile market, such as the Boston office market.

Lease options where the exercise price is a direct function of the market price at time of expiration are common and they are not difficult to price, given appropriate data from which an expected return and volatility can be estimated. Of greater interest, and certainly degree of difficulty, is the pricing of purchase and renewal options that are tied to an index, such as the CPI. Such options are commonplace in lease contracts, but there seems to have been no attempt to develop pricing models for lease options with this feature. The reason for this void is likely due to the complexity that the dynamic strike price adds to the pricing problem. When both the asset price and the strike price follow a stochastic process the Partial Differential Equation (PDE) that models the situation is significantly more complex and more difficult to solve than when only the asset price is stochastic.

In the following section, a number of PDEs that, depending on the assumed stochastic process, model the situation where the lessee is granted the right to renew the underlying lease at a rent indexed to the CPI are presented. Note that market price can be substituted for market rent in each of these equations and the right to purchase the property at a price indexed to the CPI will be modeled instead. Also, note that any index can be used as long as it exhibits similar stochastic properties to the CPI.[2]

Since it is impossible to identify the analytic solutions to the PDEs, a numerical approach will be employed to obtain approximations to the equations. A number of studies, Brennan and Schwartz (1977), Geske and Shastri (1985), Courtadon (1982), Hull and White (1990) and Hilliard (1994) have demonstrated the usefulness of the finite difference method (FDM) for approximating the solution of a PDE where the analytic solution cannot be identified. Buetow and Sochacki (1995) use a modified version of the FDM to evaluate problems similar to those addressed in this study. The FDM is used here to approximate the PDEs that are developed for pricing the lease option with a dynamic strike price.

FDM allows various dynamics to be easily incorporated into the problem and several possible stochastic processes can be used. For example, GBM can be used if it accurately represents the dynamics of the variable. Alternatively, the dynamics can be modeled by a mean reverting process (MRP) if the variable follows a trend, but experiences short term disturbances. In this study, both possibilities, as well as combinations of the two, are presented.[3]

For a solution to exist using the traditional FDM, pre-determined boundary conditions are required. When the state variables (market rent and the CPI) are extremely volatile, the solutions to these standard contingent claim finite difference equations (FDEs) are unreliable. However, this problem can be eliminated by using the absorbing boundaries (AB) technique (Sochacki, Kubichek, George, Fletcher and Smithson, 1987), which is employed in this study.

The results show that the empirical properties of the variables involved substantially impact the value of the renewal option. For example, as the market rent and the CPI become more closely related, the value of the option decreases and increases as the relationship diminishes. Several other relationships are found between the option value and the two variables and it would be expected that real estate property and portfolio managers will find these relationships to be useful in negotiating property leases.

The Model

Let $O(R,X,t)$ denote the option which gives the lessee the right to renew the lease at an indexed rent (X) at expiration $t = T$. This would be the classical Black-Scholes European call option if X were fixed. However, the options addressed here have a stochastic X resulting in dynamic boundary conditions. Due to the dynamic boundary conditions, an analytic solution is not known.

The development of the model begins by letting X follow a mean reverting process defined as:

$$dX = k_x(\mu_x - X)dt + \sigma_x X^{\gamma_X} dZ_X, \tag{3}$$

where k_x is the speed of adjustment parameter, μ_x is the long-run mean return of X, σ_x is the volatility of the returns on X, γ_X is the volatility exponent of X, and dZ_x is the standard Wiener process.[4] The square root mean-reverting process is defined when $\gamma_X = .5$. Mean-reverting processes are appropriate for positive economic variables that tend toward a long-run mean (with or without a trend) but experience short-term disturbances. Consequently, it is often used to model interest rates (Cox, Ingersoll and Ross, 1985) and the CPI, which is why an MRP model is included in addition to the GBM model.

Care must be taken when choosing the process to describe the dynamics of R, since the process must allow for $R > X$. Let the dynamics of R be expressed as follows:

$$dR = k_R(\mu_R - R)dt + \sigma_R R^{\gamma_R} dZ_R, \tag{4}$$

where the R subscript denotes the same variables defined for X to be operating on R. The values of γ_R, k_R, γ_X and k_x must allow for the possibility of $R > X$. For example, if $k_R < k_x$ and $\gamma_R < \gamma_X$ then $R > X$ for some period of time following the departure from the mean (*i.e.*, the reversion back to the mean will be slower for R than for X, thus allowing $O(R,X) > 0$). Several combinations result in positive option values.

The alternative case would be for R and X to follow GBM with the stochastic process of X defined as:

$$dX = \mu_x X dt + \sigma_x X dZ_x, \tag{5}$$

and the stochastic process of R as:

$$dR = \mu_R R dt + \sigma_R R dZ_R, \tag{6}$$

where the variables are the same as above. Again, $R > X$ must be possible.

Combinations of these processes are also possible. R can follow a GBM and X an MRP; or R an MRP and X a GBM. The dynamics chosen are dictated by the properties of the option being valued.

The PDEs

Using the usual no arbitrage assumption and a variant of the riskless-hedge portfolio, the following PDE is derived when R and X are assumed to follow the stochastic process expressed by the stochastic differential equations (SDEs) in Equations (3) and (4):

$$\frac{\sigma_R^2 R^{2\gamma_R}}{2} O_{RR} + \frac{\sigma_x^2 X^{2\gamma_x}}{2} O_{xx} + \rho_{Rx}\sigma_R\sigma_x R^{\gamma_R} X^{\gamma_x} O_{Rx} + rRO_R + rXO_x - O_\tau - rO = 0, \quad (7)$$

where $\tau = T - t$ and is the time to expiration of the option. The subscripts on O represent partial derivatives. Similarly, when both R and X follow the SDEs expressed by Equations (5) and (6) respectively, the PDE is:

$$\frac{R^2\sigma_R^2}{2} O_{RR} + \frac{X^2\sigma_x^2}{2} O_{xx} + \rho_{Rx}\sigma_R\sigma_x RXO_{Rx} + rRO_R + rXO_x - O_\tau - rO = 0. \quad (8)$$

Equation (8) is similar to Stulz (1982), except that the boundary conditions differ considerably. This difference makes the use of risk-neutral valuation an impossibility.

Equation (9) represents the PDE when R follows a GBM (Equation (6)) and X an MRP (Equation (3)):

$$\frac{R^2\sigma_R^2}{2} O_{RR} + \frac{\sigma_x^2 X^{2\gamma_x}}{2} O_{xx} + \rho_{Rx}\sigma_R\sigma_x RX^{\gamma_x} O_{Rx} + rRO_R + rXO_x - O_\tau - rO = 0. \quad (9)$$

Equation (10) is the PDE when R follows an MRP (Equation (4)) and X follows a GBM (Equation (5)):

$$\frac{\sigma_R^2 R^{2\gamma_R}}{2} O_{RR} + \frac{X^2\sigma_x^2}{2} O_{xx} + \rho_{Rx}\sigma_R\sigma_x R^{\gamma_R} XO_{Rx} + rRO_R + rXO_x - O_\tau - rO = 0. \quad (10)$$

Four PDEs (Equations (7)–(10)) have been identified that, when appropriately solved, yield the value of the option to renew the lease at a rent indexed to the CPI. If R is assumed to be the market price of the asset, instead of market rent, the same equations can be solved for the value of an option to purchase the leased space at a price indexed to the CPI.

Analytic solutions for these PDEs are not known because of the dynamics of the equations and the boundary conditions. The FDM with absorbing boundaries will be used here to approximate the solutions.[5] Since the dynamics of the strike and market rent are similar, option values are estimated using Equation (8). However, the FDM used on Equation (8) is also directly applicable to Equations (7), (9) and (10) as well.

The Boundary Conditions

In the example, the tenant has purchased an option to renew the rental agreement at a rate per square foot tied to the CPI in the following manner:

$$R_0 \times \left(1 + \frac{CPI_{t+1} - CPI_t}{CPI_t}\right), \tag{11}$$

where t represents time. At the expiration of the initial lease, the tenant has the right to renew the lease at the base rent (R_0) times the percentage change in the CPI.[6]

If the CPI increases over any given period, then the renewal rent also increases. The only way this option will take on value is if market rents (R) move differently than the CPI. That is, if the CPI increases, then it must be possible for R to increase by more than the CPI. If this is not so, and the buyer paid any amount for the option, then the no arbitrage requirement is violated.[7]

Though PDEs using both MRP and GBM stochastic processes as well as combinations of the two have been developed, the value of the renewal option for twenty-five market areas will be estimated with the assumption that real estate rents and the CPI evolve according to a GBM process. This combination is modeled by Equation (8) for which we must identify the appropriate boundary conditions.

The boundary conditions for Equation (8) when $O(R,X)$ is a call option are identified in Exhibit 1 where TV_i represents the time value of the option and the '*' denotes a value along the boundary. The AB technique enables the time value to be approximated directly from the dynamics of the PDE.

The Data

Quarterly data on the annual per square foot rent of office space are taken from the Market History Reports of the National Real Estate Index (NREI) for fourth quarter 1985 through fourth quarter 1994. These rents reflect the mean effective gross rent

Exhibit 1
Boundary Conditions for Equation 8

$\tau = 0$	$\tau > 0$
$R < X,\ O(R,X) = 0$	$O(0,X^*) = 0$
$R \geq X,\ O(R,X) = R - X$	$R^* \geq X^*,\ O(R^*,X^*) = TV_1 + R^* - X^*$
	$R^* < X^*,\ O(R^*,X^*) = TV_2$

for the market area. As would be expected, the rent patterns are similar for most markets with rents increasing over the first part of the period, then declining to a relatively static state over the second part. The primary difference across markets is the quarter the rent peaked. The data is used to obtain an estimate of the historical volatility of market rent for each of the twenty-five market areas reported by NREI as well as the national market. These volatility estimates are then used with other parameter estimates to solve Equation (8) for the value of an option to renew a five-year lease for an additional five years at a contract rent as defined by Equation (11).

Results

Exhibit 2 presents the value of the option to renew at the indexed rent stated as an increment to the annual rent that would exist in the absence of such an option. For example, in the Boston office market, which according to the data had the most volatile market rent during the data period, a lessor should be indifferent between a lease with a base rent per square foot of R_B per year and no option to renew the lease other than at market, and a lease with a base rent of (R_B + \$1.03) per year with an option to renew at R_B grossed up by the cumulative change in the CPI over the initial five-year lease period.[8] Similarly, a lessor in the northern New Jersey market, which had the most stable rent over the data period, would be indifferent between R_{NJ} and R_{NJ} + \$.72. The renewal option is worth \$.31/yr. more in per square foot rent in the volatile Boston market than in the relatively stable northern New Jersey market. It is interesting that while the rent volatility differs widely across markets, from the northern New Jersey low of .019 to the Boston high of .0749, the value of the market specific options differ by a relatively modest \$.31/yr. in a five-year rental stream. The highest volatility is four times as great as the lowest, yet the value of the option for the high volatility is only 43% greater than the value of the option for the low volatility. Because of the cross correlation term in the PDE, the impact of rent

Exhibit 2
Value of Five-Year Renewal Options–\$/s.f.

Location	σ_R	Option(\$)	Location	σ_R	Option(\$)
Atlanta	0.0305	0.75	NYC-mdtwn	0.0475	0.82
Baltimore	0.0413	0.80	Orange Co.	0.0580	0.88
Boston	0.0749	1.03	Orlando	0.0330	0.77
Charlotte	0.0474	0.83	Philadelphia	0.0280	0.75
Chicago	0.0627	0.90	Phoenix	0.0531	0.86
Dallas/FX	0.0263	0.73	Riverside	0.0461	0.80
Denver	0.0382	0.77	San Diego	0.0514	0.86
Houston	0.0359	0.75	San Francisco	0.0372	0.79
Los Angeles	0.0522	0.86	Sacramento	0.0377	0.77
Miami	0.0450	0.81	Seattle	0.0353	0.76
Minn/SP	0.0560	0.86	Tampa/SP	0.0395	0.78
NJ-North	0.0190	0.72	Washington, DC	0.0643	0.94
NYC-dntwn	0.0341	0.76	National	0.0290	0.74

volatility is more significant in some market areas than in others as illustrated earlier. However, despite the correlation term the option value remains monotonic in volatility.

The average value of the renewal options for all markets is \$.81, with a standard deviation of \$.072, and all but two values lie within a \$.10/sq. ft./yr. range of this average. Indeed, this average value would not represent a gross error for an estimate of the value of the option in any market area. It is reasonable to expect that the within market variance of option values would be significantly less than the across market variance, and the option values estimated here from aggregated data for each market would be a close estimate of the option value for a particular property within a specific market area. Therefore, it is not necessary that the lessor and lessee in every transaction estimate the value of the renewal option since the mean value estimated from the aggregated data should be a very close approximation of the property-specific value.

Several variables, other than volatility, affect the relative values of the embedded renewal options. The larger the initial value of *R*, the more valuable the option. The more negatively (or less positively) correllated *R* and *X*, the more valuable the option. If *R* and *X* move in the opposite direction, then the limited downside of the embedded call has greater value. A complete examination of these variables, and their relative impact on the value of the option, is beyond the scope of this article, but is an area ripe for future research.

Since detached market prices of embedded lease options do not exist, it is impossible to test the robustness of the estimates of this study against the market. However, it is possible to verify the model intuitively by comparing the properties of the model to those implied by basic option theory. These properties, referred to in the option literature as the ‘‘Greeks,’’ have a priori signs and those of the model should conform to these signs.[9] Exhibit 3 presents the eight Greeks of the model and the expected sign on each. The last column indicates whether the expected sign was observed from the model. The answer in all cases is yes, which attests to the theoretical correctness of the model. It also provides the basis for hedging the option with standard risk management strategies.

Since for less complicated call options an increase in time to maturity (*theta*) as well as an increase in volatility (*kappa* or *vega*) would cause the option to take on greater value, the indeterminant signs on *theta* and *kappa* (*X*) require some explanation. Since the options examined here have a dynamic strike price, more volatility in the strike and more time to expiration do not necessarily add value to the option. The volatility of *X* has a varying effect on the value of the option depending on the correlation between *R* and *X*. For both at and out-of-the money options, both *theta* and *kappa* (*X*) have the expected positive sign, but for in-the-money options, both signs can be negative under certain scenarios. This is primarily due to the cross derivative term in Equation (8) and the interrelationship between *R* and *X*. As *R* and *X* become more perfectly positively correlated time value can become slightly negative. When the two are perfectly correlated, the dynamics of the option are such that, the value attached

Exhibit 3
The Greeks

Greek Name	Mathematical Equivalent	Expected Effect	Verified (Y/N)
Delta (*r*)	$\frac{\partial O}{\partial R}$	+	Y
Gamma (*r*)	$\frac{\partial^2 O}{\partial R^2}$	+	Y
Theta	$\frac{\partial O}{\partial \tau}$	+/−	Y
Kappa (*r*), *Vega* (*r*)	$\frac{\partial O}{\partial \sigma_R}$	+	Y
Rho	$\frac{\partial O}{\partial O}$	+	Y
Delta (*x*)	$\frac{\partial O}{\partial X}$	−	Y
Gamma (x)	$\frac{\partial^2 O}{\partial X^2}$	+	Y
Kappa (*x*), *Vega* (*x*)	$\frac{\partial O}{\partial \sigma_x}$	+/−	Y

to the likelihood of an increase in intrinsic value is less than the financing cost for the option over its life. Therefore, it is possible for both *theta* and *kappa* (X) to be negative under special conditions.

Conclusion

This study shows that it is possible to estimate the value of embedded options in lease contracts that give the lessee the right to renew the lease or purchase the property at a rent or price tied to the cumulative change in some index such as the CPI. Additionally, it has been demonstrated how the value of renewal or purchase options with non-indexed strikes could be estimated. Since both types of options are commonplace in lease contracts, this capability should have considerable value to property/asset managers charged with negotiating the most favorable lease on behalf of the property investor.

The approach to option valuation taken here could be employed in a broad array of contingent claims in real estate transactions. The option to purchase a property at a particular cap rate could be similarly modeled. Any purchase option with a fixed strike price could be evaluated by dropping the index component on R. The model is also applicable to pricing the standard indexed lease, where an initial base rent is adjusted periodically according to the change in the CPI, if, as is most commonly the case, the base rent is a lower bound. While the value of indexation has been explored in the absence of the lower bound, this model would incorporate the contingent nature which the lower bound attaches to an indexed rent and provide a more exact value of the indexation.[10]

An obvious question that arises from this analysis is the reliability of aggregated rental data. It has been specifically shown that renewal options in real estate leases have significant value, as do purchase options, and any other option that conveys a valuable right to either the lessee or lessor. If rental data are aggregated without adjusting for the value of embedded options, it will be reliable only if all leases, to which the rents attach, are uniform with respect to the embedded options. In addition to offering option values to market participants, this study also provides a mechanism for adjusting rent data for greater uniformity and reliability. This tangential benefit is extraordinarily important.

The complexity of the option-pricing model presented here probably precludes its' use to individual properties, but the option value for a market area would be a close estimate for a property within that market. These values could be provided on a continuing basis by one or more of the real estate data services. This would allow a broader menu of lease options to be negotiated by individual lessees and lessors without the necessity of having an in house pricing capability. With this generality the model should find wide application as a tool for lease negotiation.

Appendix

This appendix develops the explicit FDEs to approximate Equations (7) and (8) and discusses the concept of AB as it applies to the FDM. Stability requirements are well known in the mathematics literature (Hoffman, 1992) and will not be repeated here.

Define the following variables:

r = The risk free interest rate;
ΔX = The increment used for the strike price;
ΔrR = The increment used for the underlying asset;
$\Delta tauR$ = The time to expiration increment; and
$O^n_{i,j}$ = The value of the option at time step n, asset step i and strike price step j.

Note that expiration is denoted by $n = 0$, and all subsequent $n > 0$ are actually going backwards in calendar time. Therefore, the FDE for Equation (7) is as follows:

$$O^{n+1}_{i,j} = AO^n_{i,j} + B[O^n_{i+1,j} - O^n_{i,j}] + C[O^n_{i,j+1} - O^n_{i,j}] + D[O^n_{i+1,j} + O^n_{i-1,j} - 2O^n_{i,j}] + E[O^n_{i,j+1} + O^n_{i,j-1} - 2O^n_{i,j}] + F[O^n_{i+1,j+1} - O^n_{i+1,j-1} - O^n_{i-1,j+1} + O^n_{i-1,j-1}], \quad (7')$$

where:

$$A = 1 - r\Delta\tau, \qquad B = rR\,\frac{\Delta\tau}{\Delta R}, \qquad C = rX\,\frac{\Delta\tau}{\Delta X}, \qquad D = \frac{\Delta\tau}{(\Delta R)^2}\,\frac{R^{2\gamma_R}\sigma_R^2}{2},$$

$$E = \frac{\Delta\tau}{(\Delta X)^2}\,\frac{X^{2\gamma_X}\sigma_X^2}{2} \qquad \text{and} \qquad F = \frac{\Delta\tau}{4(\Delta R)(\Delta X)}\,R^{\gamma_R}\,X^{\gamma_X}\,\sigma_R\,\sigma_X\,\rho_{RX}.$$

Similarly for Equation (8):

$$O_{i,j}^{n+1} = A'O_{i,j}^{n} + B'[O_{i+1,j}^{n} - O_{i,j}^{n}] + C'[O_{i,j+1}^{n} - O_{i,j}^{n}] + D'[O_{i+1,j}^{n} + O_{i-1,j}^{n} - 2O_{i,j}^{n}]$$
$$+ E'[O_{i,j+1}^{n} + O_{i,j-1}^{n} - 2O_{i,j}^{n}] + F'[O_{i+1,j+1}^{n} - O_{i+1,j-1}^{n} - O_{i-1,j+1}^{n} + O_{i-1,j-1}^{n}], \quad (8')$$

where:

$$A' = 1 - r\Delta\tau, \qquad B' = rR\frac{\Delta\tau}{\Delta R}, \qquad C' = rX\frac{\Delta\tau}{\Delta X}, \qquad D' = \frac{\Delta\tau}{(\Delta R)^2}\frac{R^2\sigma_R^2}{2},$$

$$E' = \frac{\Delta\tau}{(\Delta X)^2}\frac{X^2\gamma_X\sigma_X^2}{2} \qquad \text{and} \qquad F' = \frac{\Delta\tau}{4(\Delta R)(\Delta X)} R\ X\ \sigma_R\ \sigma_X\ \rho_{RX}.$$

Applying FDM directly to these equations would encounter inaccuracies when the volatilities of R or X are too high. This is a result of the pre-determined boundary conditions required for the implementation of FDM. Without known boundary conditions unlimited memory is necessary to solve the problem using FDM. Absorbing boundaries as discussed in Sochacki, Kubichek, George, Fletcher and Smithson (1987), allow a way around this obstacle. Additionally, this technique enables the time value to be better approximated.

The technique involves defining two concentric regions of computation on our FDM grid. The inner region is the area of primary concern. It is within this region that the option values are desired. The outer region acts as a buffer or sponge-like region that decays the solution slowly so that a model of infinite extent is approximated. The decay of the solution is done by the use of an exponential damper of the form, $A(S,X)O(S,X,t)$, where $A(S,X)$ is defined in Sochacki, et al. (1987).

This process allows the boundaries of the inner region to change according to the dynamics of the PDE, thus allowing for accurate approximations for the option values. It also yields approximations to the time value along the boundaries of the inner or primary region. Standard FDM does not offer this result.

Endnotes

[1]See, for example, Newman (1991) and Posner (1992).
[2]The operating cost indexes of BOMA and IREM, for example, could be used instead of the CPI. Since the primary determinant of a cost index is the change in the general price level the dynamics should be the same as the CPI.
[3]We present estimates of the value of the renewal option for twenty-five market areas using the assumption that both market rent and the CPI follow a GBM process. However, we develop the PDEs for several possible combinations of stochastic processes and a comparison of the estimates provided by these PDEs is a project for future research.
[4]See Friedman (1975, Chap. 3) for an excellent explanation of stochastic processes.
[5]The use of AB in the finite difference method is discussed in the Appendix.
[6]We assume an initial five-year lease at a fixed rent of R_O and a renewal option for an additional five years.
[7]We note that it only must be possible for market rents to move differently than the CPI. It is not a requirement of the model that at some point they must.

[8]The subsequent five year lease is assumed to be free of an option to renew. If such an option is to be included then the base rent that would be grossed up is (R_B + $1.03).
[9]Greeks is a term used by practicing professionals that refers to the change in the option's value given a small change in the value of the model's variables (*i.e.*, the various partial derivatives).
[10]Albert and MacIntosh (1989) price the value of indexation when there is no lower bound.

References

Albert, J. and W. MacIntosh, Identifying Risk Adjusted Indifference Rents for Alternative Operating Leases, *Journal of Real Estate Research*, 1989, 4:3, 22–36.

Black, F. and M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 1973, 83, 617–37.

Brennan, M. and E. Schwartz, The Valuation of American Put Options, *Journal of Finance*, 1977, 32, 449–462.

Buetow, G. and J. Sochacki, A Finite Difference Approach to the Pricing of Options Using Absorbing Boundaries, *Journal of Financial Engineering*, 1995, 263–80.

Courtadon, G., A More Accurate Finite Difference Approximation for the Valuation of Options, *Journal of Financial and Quantitative Analysis*, 1982, 697–703.

Cox, J. C., J. E. Ingersoll and S. A. Ross, A Theory of the Term Structure of Interest Rates, *Econometrica*, 1985, 53, 385–407.

Friedman, A., Stochastic Differential Equations and Applications, New York: Academic Press, 1975.

Geske, R. and K. Shastri, Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques, *Journal of Financial and Quantitative Analysis*, 1985, 45–71.

Hilliard, J. E., Finite Horizon Hedge Ratios for American Options: A Minimum Variance Solution, *Journal of Financial Engineering*, 1994, 1–17.

Hoffman, J., Numerical Methods for Engineers and Scientists, New York: McGraw Hill, 1992.

Hull, J. and A. White, Valuing Derivative Securities Using the Explicit Finite Difference Method, *Journal of Financial and Quantitative Analysis*, 1990, 87–99.

Newman J., The Option Period B Is It a One-Way Street Called ''Tenant Avenue?'' *Real Estate Review*, 1991, Fall, 77–81.

Posner K., Hedging Against Changes in Market Rent Levels, *Real Estate Review*, 1992, 92, 21–26.

Sochacki, J., R. Kubichek, J. George, W. Fletcher and S. Smithson, Absorbing Boundary Conditions and Surface Waves, *Geophysics*, 1987, 52, 60–71.

Stulz, R., Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications, *Journal of Financial Economics*, 1982, 10, 161–185.